ELSEVIER
Linear Algebra and its Applications 350 (2002) 147–170
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

The polytope of degree sequences of hypergraphs

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Received 15 August 2000; accepted 4 January 2002

Submitted by R.A. Brualdi

Abstract

Let $D_n(r)$ denote the convex hull of degree sequences of simple r-uniform hypergraphs on the vertex set $\{1, 2, \ldots, n\}$. The polytope $D_n(2)$ is a well-studied object. Its extreme points are the threshold sequences (i.e., degree sequences of threshold graphs) and its facets are given by the Erdös–Gallai inequalities. In this paper we study the polytopes $D_n(r)$ and obtain some partial information. Our approach also yields new, simple proofs of some basic results on $D_n(2)$. Our main results concern the extreme points and facets of $D_n(r)$. We characterize adjacency of extreme points of $D_n(r)$ and, in the case $r = 2$, determine the distance between two given vertices in the graph of $D_n(2)$. We give a characterization of when a linear inequality determines a facet of $D_n(r)$ and use it to bound the sizes of the coefficients appearing in the facet defining inequalities; give a new short proof for the facets of $D_n(2)$; find an explicit family of Erdös–Gallai type facets of $D_n(r)$; and describe a simple lifting procedure that produces a facet of $D_{n+1}(r)$ from one of $D_n(r)$. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: Primary 05C07; Secondary 05C65, 90C10, 90C27

Keywords: Degree sequences; Hypergraphs; Polytopes; Threshold graphs; Threshold sequences

1. Introduction

The problem of characterizing degree sequences of simple uniform hypergraphs is a long standing open question in hypergraph theory (see [3]). Though a complete characterization is not known, several partial results are available (see [1,3,4,6,8,9]). In this paper we study an object that is closely related to this problem, namely, the

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PII: S0024-3795(02)00272-0

polytope of degree sequences. To motivate our study we first consider the case of graphs. All graphs and hypergraphs considered in this paper are simple and uniform and (most of them) have vertex sets equal to $[n] = \{1, 2, \ldots, n\}$, for some positive integer n.

The *degree sequence* of a graph $G = ([n], E)$ is the nonnegative integral vector $d_G = (d_1, \ldots, d_n)$, where d_i is the degree of vertex i. The *polytope of degree sequences* is defined by $D_n =$ convex hull $\{d_G\colon\ G \text{ is a graph on } [n]\}$. We have:

Theorem 1.1. *A nonnegative integral vector $d = (d_1, d_2, \ldots, d_n)$ is a degree sequence of a graph if and only if $d \in D_n$ and $\sum_{i=1}^{n} d_i$ is even.*

The proof of Theorem 1.1 proceeds in two steps. The first step is the following classical result, due to Erdös and Gallai [10].

Theorem 1.2. *A weakly decreasing sequence $(d_1, d_2, \ldots, d_n)$ (i.e., $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$) of nonnegative integers is the degree sequence of a graph if and only if*

$$\sum_{i=1}^{n} d_i \text{ is even}, \tag{1}$$

$$\sum_{i=1}^{j} d_i - j(j-1) \leqslant \sum_{k=j+1}^{n} \min(j, d_k), \qquad j = 1, 2, \ldots, n. \tag{2}$$

For every $j = 1, 2, \ldots, n$, and $l = j, \ldots, n$, the r.h.s of (2) is clearly $\leqslant \sum_{k=j+1}^{l} j + \sum_{k=l+1}^{n} d_k = (l-j)j + \sum_{k=l+1}^{n} d_k$. Moreover, under the assumption $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$, the system (2) of n nonlinear inequalities (in the d_i's) is equivalent to the following system of $n(n+1)/2$ linear inequalities:

$$\sum_{i=1}^{j} d_i - j(j-1) \leqslant (l-j)j + \sum_{k=l+1}^{n} d_k, \quad j = 1, 2, \ldots, n, \ \ l = j, \ldots, n.$$

It is convenient to write the above system of linear inequalities as

$$\sum_{i=1}^{j} d_i - \sum_{k=l+1}^{n} d_k \leqslant j(n-1-(n-l)), \quad j = 1, 2, \ldots, n, \ \ l = j, \ldots, n. \tag{3}$$

Thus, a weakly decreasing sequence $(d_1, d_2, \ldots, d_n)$ of nonnegative integers is the degree sequence of a graph if and only if (1) and (3) are satisfied.

Now suppose we are considering arbitrary degree sequences and not just those that are weakly decreasing. In (3) we were taking sums of the j largest and $n-l$

smallest degrees. We shall replace those by arbitrary sums. Define a polytope P_n in $\mathbb{R}^n$, as the solution set of the following system of linear inequalities:

$$\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant |S|(n-1-|T|) \quad \text{for all sets } S, T \subseteq [n], \quad S\cap T=\emptyset. \tag{4}$$

(Taking $T=\{i\},\ S=\emptyset$ gives $x_i \geqslant 0$, and taking $S=\{i\},\ T=\emptyset$ gives $x_i \leqslant n-1$. This shows that P_n is indeed a polytope.)

By Theorem 1.2, every degree sequence (not necessarily weakly decreasing) satisfies the system (4) and thus $D_n \subseteq P_n$. The second step in the proof of Theorem 1.1 is the following result, due to Koren [13] and Peled and Srinivasan [16], which shows that (4) gives a linear inequality description of D_n.

Theorem 1.3. $D_n = P_n$.

It is now easily seen that Theorem 1.1 is an immediate consequence of Theorems 1.2 and 1.3. Koren proved Theorem 1.3 by showing that every extreme point of P_n is a degree sequence. Moreover, he characterized the extreme points of D_n as exactly the degree sequences that have a unique realization. (A degree sequence $a=(a_1,a_2,\ldots,a_n)$ is *uniquely realizable* provided the following condition holds: suppose $G=([n],E)$ and $G'=([n],E')$ satisfy $a=d_G=d_{G'}$. Then $E=E'$.)

Chvátal and Hammer [5] introduced an important class of graphs, namely, the *threshold graphs*. Threshold graphs have numerous characterizations (see [15]), one of which says that they are precisely the graphs with uniquely realizable degree sequences. Thus the extreme points of D_n are precisely the *threshold sequences*, i.e., degree sequences of threshold graphs. Using linear programming and the structure of threshold graphs a detailed study of D_n was carried out in [16], yielding a new proof of Theorem 1.3. Adjacency of extreme points, facets of D_n, and majorization among degree sequences were also treated in [16] (also see [2]).

A (*simple*) *r-uniform hypergraph* is a pair $([n],E)$, where $E\subseteq\binom{[n]}{r}$ (= set of all r-element subsets of $[n]$). The elements of $[n]$ are called *vertices* and the elements of E are called *edges*. For a vertex $i\in[n]$, the degree $d_i=d_H(i)$ of i is the number of edges containing the vertex i. The sequence $d=d_H=(d_1,d_2,\ldots,d_n)$ is called the *degree sequence* of H. Simple r-uniform hypergraphs will henceforth be called r-hypergraphs. Define *the polytope of degree sequences of r-hypergraphs* by $D_n(r)=$ convex hull $\{d_H\colon\ H$ is an r-hypergraph on $[n]\}$. (Note that $D_n(2)=D_n$ as defined above.)

The main purpose of this paper is to study the polytopes $D_n(r)$ and obtain some partial information. Our approach also yields new and simple proofs of some basic results on $D_n(2)$. Section 2 treats extreme points of $D_n(r)$. We characterize adjacency of extreme points of $D_n(r)$ and, in the case $r=2$, determine the distance between two given vertices in the graph of $D_n(2)$. We also answer in the negative a question by Golumbic [11] on whether a theorem of Chvátal and Hammer on threshold graphs generalizes to the hypergraph case. In Section 3 we study facets of $D_n(r)$. We first give a characterization of when a linear inequality determines a facet of $D_n(r)$ and use this to bound the sizes of the coefficients appearing in the facet determining inequalities. The problem of finding the facets of $D_n(r)$ then reduces to the problem of finding linear inequalities satisfying this characterization. This can actually be carried out in the case $r = 2$ and leads to a new, simple proof for the facets of $D_n(2)$. We also give an explicit family of Erdös–Gallai type facets of $D_n(r)$. We prove a simple lifting theorem, whereby facets of $D_n(r)$ can be lifted to produce facets of $D_{n+1}(r)$ and, using this, we give a facet of $D_n(3)$ where the coefficients of the l.h.s are the Fibonacci numbers. Section 4 collects results on the ranks of various incidence matrices which are needed in Section 3 (all ranks in this paper are over the reals). Finally, in Section 5, we list the facets of $D_4(3)$, $D_5(3)$ and $D_6(3)$, found using the package PORTA [7], due to Christof.

2. Extreme points

In this section we study the extreme points of the polytope $D_n(r)$. Given an n-tuple $c = (c_1, c_2, \ldots, c_n)$ of real numbers and $X \subseteq [n]$ we shall denote the sum $\sum_{i\in X} c_i$ by $c(X)$.

Definition 2.1. An r-hypergraph $H = ([n], E)$ is called an *r-threshold hypergraph* if there exists an n-tuple $c = (c_1, c_2, \ldots, c_n)$ of real numbers such that for all $X \in \binom{[n]}{r}$, we have $X \in E$ if and only if $c(X) \geqslant 0$. The c_i's are called *weights* and we say that H is determined by the weights $(c_1, c_2, \ldots, c_n)$. The degree sequence of an r-threshold hypergraph is called an *r-threshold sequence*.

In the case $r = 2$, Definition 2.1 reduces to one of the many characterizations of threshold graphs [15]. In [16], the polytope $D_n(2)$ was studied using the structure theorem for threshold graphs. It turns out (as will be shown in the present paper) that the analysis of $D_n(2)$ becomes considerably simpler if one adopts Definition 2.1 as the starting point.

Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$. Consider the following combinatorial optimization problem:

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad & (x_1, x_2, \ldots, x_n) \in D_n(r). \end{aligned}$$

The following simple algorithm produces an optimal solution. For fixed r, its running time is clearly polynomial in n.

Algorithm 2.2.

$E \leftarrow \emptyset$

for $\left(\text{all } X \in \binom{[n]}{r}\right)$ do

if $c(X) \geqslant 0$ then $E \leftarrow E \cup \{X\}$

$(x_1, x_2, \ldots, x_n) \leftarrow$ degree sequence of $([n], E)$.

Theorem 2.3. *Algorithm* 2.2 *is correct.*

Proof. Let $DS_n(r)$ denote the set of all degree sequences of r-hypergraphs on $[n]$. The set of extreme points of $D_n(r)$ is a subset of $DS_n(r)$. Thus, maximizing the linear function $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ over $D_n(r)$ is equivalent to maximizing it over the finite set $DS_n(r)$.

Let $([n], E)$ be an r-hypergraph with degree sequence $(d_1, d_2, \ldots, d_n) \in DS_n(r)$. Since each edge in E contributes 1 to the degree of r vertices (and 0 to the other $n - r$ vertices), we have

$$c_1d_1 + c_2d_2 + \cdots + c_nd_n = \sum_{X \in E} c(X).$$

It is now clear that the algorithm is correct. $\square$

Remark 2.4. The proof of Theorem 2.3 actually shows that, for $H = ([n], E)$, d_H is an optimal degree sequence if and only if $E = A \coprod C$ ($\coprod$ denotes disjoint union), where $A = \{X \subseteq [n]\colon\ c(X) > 0\}$ and $C \subseteq D = \{X \subseteq [n]\colon\ c(X) = 0\}$. Thus, if D is empty, there is one and only one hypergraph with optimal degree sequence.

Theorem 2.5. *A degree sequence is an extreme point of $D_n(r)$ if and only if it is an r-threshold sequence.*

Proof. (*Only if*) Every extreme point of $D_n(r)$ is the unique maximum for some objective function $\sum_{i=1}^{n} c_ix_i$, so Algorithm 2.2 must produce it on input $(c_1, c_2, \ldots, c_n)$. Clearly the algorithm only produces r-threshold hypergraphs and thus the extreme points of $D_n(r)$ are r-threshold sequences.

(*If*) Let $a = (a_1, a_2, \ldots, a_n)$ be an r-threshold sequence and $H = ([n], E)$ an r-threshold hypergraph with $d_H = a$. Let $c = (c_1, c_2, \ldots, c_n)$ be weights determining H. Let $\epsilon = -\max\{c(X)\colon\ X \in \binom{[n]}{r} - E\}$. Note that $\epsilon > 0$. Choose a positive δ such that $r\delta < \epsilon$. Define $(c'_1, c'_2, \ldots, c'_n)$ by $c'_i = c_i + \delta$ for all i.

Now, clearly on input $c' = (c'_1, c'_2, \ldots, c'_n)$, Algorithm 2.2 outputs H and its degree sequence a. Moreover, since $c'(X) \neq 0$ for all r-subsets $X \in \binom{[n]}{r}$, we have by Remark 2.4 that a is the unique optimal degree sequence. Thus a is an extreme point of $D_n(r)$. $\square$

Corollary 2.6. *Let $a = (a_1, a_2, \dots, a_n)$ be an r-threshold sequence. If $H = ([n], E)$ and $H' = ([n], E')$ are r-hypergraphs with $a = d_H = d_{H'}$, then $H = H'$, i.e., r-threshold sequences are uniquely realizable.*

Proof. By Theorem 2.5, a is an extreme point of $D_n(r)$ and is thus a unique maximum for some objective function, say $\sum_{i=1}^n c_i x_i$. By Remark 2.4, $c(X) \neq 0$ for all $X \in \binom{[n]}{r}$ (here $c = (c_1, c_2, \dots, c_n)$) and hence there is a unique r-hypergraph with degree sequence a. Thus $H = H'$. $\square$

As mentioned in Section 1, the converse of Corollary 2.6 is also true in the case of graphs. We do not know whether the converse continues to hold for $r \geqslant 3$.

Given an r-threshold hypergraph $H = ([n], E)$, determined by the weights $c = (c_1, c_2, \dots, c_n)$, we define the weight of an r-subset $X \in \binom{[n]}{r}$ to be $c(X)$.

Lemma 2.7. *Let $H = ([n], E)$ be an r-threshold hypergraph determined by the weights $c = (c_1, \dots, c_n)$.*

(i) *Let $i \in [n]$ have nonzero degree in H. From among all the edges in E containing the vertex i, choose an edge X with minimum weight. Then $([n], E - \{X\})$ is an r-threshold hypergraph.*

(ii) *Let $j \in [n]$ have nonzero degree in the complement $([n], \binom{[n]}{r} - E)$ of H (i.e., $d_H(j) < \binom{n-1}{r-1}$). From among all r-subsets not in E and containing the vertex j, choose an r-subset X with maximum weight. Then $([n], E \cup \{X\})$ is an r-threshold hypergraph.*

Proof. We may assume that $r \geqslant 2$, as the result is trivially true when $r = 1$.

(i) Let $\epsilon = c(X)$. Define $(c'_1, c'_2, \dots, c'_n)$ by $c'_l = c_l$ if $l \neq i$ and $c'_i = c_i - \epsilon$. It is easily seen that the weights $c' = (c'_1, c'_2, \dots, c'_n)$ determine the same r-threshold hypergraph $H = ([n], E)$ as the weights c. Note that we now have $c'(X) = 0$.

Let $\epsilon_1 = -\max\{c'(Y)\colon Y \in \binom{[n]}{r} - E\}$. Note that $\epsilon_1 > 0$. Choose positive δ_1, δ_2 such that $r\delta_2 < \epsilon_1$ and $(r-1)\delta_1 \leqslant \delta_2$. Define $(d_1, d_2, \dots, d_n)$ by

$$d_l = \begin{cases} c'_i & \text{if } l = i, \\ c'_i - \delta_1 & \text{if } l \in X - \{i\}, \\ c'_i + \delta_2 & \text{if } l \in [n] - X. \end{cases}$$

We claim that $([n], E - \{X\})$ is an r-threshold hypergraph determined by the weights $d = (d_1, \dots, d_n)$. Let $Y \in \binom{[n]}{r}$. Consider the following three cases:

(a) $Y = X$: we have $d(Y) = c'(Y) - (r-1)\delta_1 = (-(r-1)\delta_1) < 0$.

(b) $Y \in \binom{[n]}{r} - E$: we have $d(Y) \leqslant c'(Y) + r\delta_2 \leqslant -\epsilon_1 + r\delta_2 < 0$.

(c) $Y \in E - \{X\}$: we must have $Y \cap ([n] - X) \neq \emptyset$. Thus $d(Y) \geqslant c'(Y) + (\delta_2 - (r-1)\delta_1) \geqslant 0$.

(ii) Let $\epsilon = -\max\{c(Y)\colon\ Y \in \binom{[n]}{r} - E\}$. Choose positive δ with $r\delta < \epsilon$. Define $c' = (c'_1, c'_2, \ldots, c'_n)$ by $c'_i = c_i + \delta$ for all i. Then the weights c' determine the same r-threshold hypergraph $H = ([n], E)$ as the weights c and $c'(Y) \neq 0$ for all $Y \in \binom{[n]}{r}$. Moreover, with respect to the weights c', X is still the maximum weight subset among all r-subsets not in E and containing j. The result now follows from part (i) by considering the complementary r-threshold hypergraph $([n], \binom{[n]}{r} - E)$, determined by the weights $(-c'_1, -c'_2, \ldots, -c'_n)$. □

Theorem 2.8. *Let f and g be two r-threshold sequences. Let* $([n], E_1)$ *and* $([n], E_2)$ *be the* (*unique*) *r-hypergraphs with degree sequences f and g*, *respectively. Then f and g are adjacent extreme points of* $D_n(r)$ *if and only if* $|E_1 \oplus E_2| = 1$.

Proof. (*If*) Let $|E_1 \oplus E_2| = 1$. Without loss of generality we may assume that $E_2 = E_1 \cup \{X\}$, where $X \in \binom{[n]}{r}$.

Let $([n], E_1)$ be determined by the weights $c = (c_1, c_2, \ldots, c_n)$. We can assume that $c(Y) > 0$ (resp. $c(Y) < 0$) for all $Y \in E_1$ (resp. $Y \notin E_1$) (this can be done as in the proof of the 'if' part of Theorem 2.5). Similarly, we can assume that $([n], E_2)$ is determined by weights $d = (d_1, d_2, \ldots, d_n)$ such that $d(Y) > 0$ (resp. $d(Y) < 0$) for all $Y \in E_2$ (resp. $Y \notin E_2$).

Let $z = d(X)$ and $y = -c(X)$. Note that $z, y > 0$. Define $a = (a_1, a_2, \ldots, a_n) = (z/y)(c_1, c_2, \ldots, c_n) + (d_1, d_2, \ldots, d_n)$.

We claim that f and g are the only two degree sequences which maximize $\sum_{i=1}^{n} a_i x_i$ over the set of all degree sequences. It will then follow that f and g are adjacent. Let $Y \in \binom{[n]}{r}$. Consider the following three cases:
(a) $Y \in E_1$: we have $a(Y) = (z/y)c(Y) + d(Y) > 0$.
(b) $Y \notin E_2$: we have $a(Y) = (z/y)c(Y) + d(Y) < 0$.
(c) $Y = X$: we have $a(Y) = (z/y)c(X) + d(X) = 0$.

It now follows from Remark 2.4 that $([n], E_1)$ and $([n], E_2)$ are the only r-hypergraphs whose degree sequences maximize $\sum_{i=1}^{n} a_i x_i$.

(*Only if*) We shall assume $|E_1 \oplus E_2| \geqslant 2$ and derive a contradiction. By the definition of adjacency there exists a $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} c_i x_i$ is maximized over the r-threshold sequences only by f and g. Algorithm 2.2 on input c will produce one of $([n], E_1)$ or $([n], E_2)$, say $([n], E_1)$. Let $X \in E_1 \oplus E_2$, where $X \in \binom{[n]}{r}$. By Remark 2.4, $c(X) = 0$ and hence $X \in E_1$. By Lemma 2.7, $([n], E_1 - \{X\})$ is r-threshold. This is a contradiction, since $([n], E_1 - \{X\})$ is different from both $([n], E_1)$ and $([n], E_2)$, whence its degree sequence h differs from both f and g by Corollary 2.6, and h maximizes $\sum_{i=1}^{n} c_i x_i$. □

The graph $G(P)$ of a polytope P has a vertex for each extreme point of P, and two vertices of $G(P)$ are connected by an edge if and only if the corresponding extreme points in P are adjacent. The *diameter* of P is the diameter of the graph $G(P)$.

Theorem 2.9. *The diameter of* $D_n(r)$ *is* $\binom{n}{r}$.

Proof. Let $H_1 = ([n], E_1)$ and $H_2 = ([n], E_2)$ be two r-threshold hypergraphs. By Lemma 2.7(i) and Theorem 2.8 there is a path in $G(D_n(r))$ from d_{H_i} $(i = 1, 2)$ to $(0, 0, \ldots, 0)$ of length $|E_i|$ $(i = 1, 2)$. Thus, if $|E_1| + |E_2| \leqslant \binom{n}{r}$, we have that the distance between d_{H_1} and d_{H_2} is $\leqslant \binom{n}{r}$.

By Lemma 2.7(ii) and Theorem 2.8 there is a path in $G(D_n(r))$ from d_{H_i} $(i = 1, 2)$ to $(\binom{n-1}{r-1}, \binom{n-1}{r-1}, \ldots, \binom{n-1}{r-1})$ of length $\binom{n}{r} - |E_i|$ $(i = 1, 2)$. Thus, if $|E_1| + |E_2| \geqslant \binom{n}{r}$, we have that the distance between d_{H_1} and d_{H_2} is $\leqslant 2\binom{n}{r} - (|E_1| + |E_2|) \leqslant \binom{n}{r}$.

It is easily verified that the distance between $(0, \ldots, 0)$ and $(\binom{n-1}{r-1}, \ldots, \binom{n-1}{r-1})$ is $\binom{n}{r}$. The result follows. $\square$

In the case of graphs, i.e., $r = 2$, we can strengthen Theorem 2.9 as follows (we do not know whether a similar statement holds for $r \geqslant 3$).

Theorem 2.10. *Let f and g be threshold sequences with (unique) realizations* $T_1 = ([n], E_1)$ *and* $T_2 = ([n], E_2)$, *respectively. Then the distance between f and g in* $G(D_n(2))$ *is* $|E_1 \oplus E_2|$.

Proof. Let T_1 (resp. T_2) be determined by the weights $(c_1, \ldots, c_n)$ (resp. $(d_1, \ldots, d_n)$). Without loss of generality we may assume $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$.

The proof is by induction on $n + |E_1 \oplus E_2|$. Consider the following two cases.

(a) *Vertex n is isolated in* T_2: If vertex n is also isolated in T_1, then we can complete the proof by induction by passing to the threshold graphs $([n-1], E_1)$ and $([n-1], E_2)$. If vertex n is not isolated in T_1, then by Lemma 2.7(i), there is an edge $e \in E_1$ containing the vertex n such that $T_3 = ([n], E_1 - \{e\})$ is threshold. Note that $|(E_1 - \{e\}) \oplus E_2| = |E_1 \oplus E_2| - 1$. Let h be the degree sequence of T_3. Then f and h are adjacent in $G(D_n(2))$ and, by induction, the distance between h and g is $|E_1 \oplus E_2| - 1$. Since clearly the distance between f and g is $\geqslant |E_1 \oplus E_2|$, the result follows.

(b) *Vertex n is not isolated in* T_2: This means that $d_i + d_n \geqslant 0$ for some i. Since $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ it now follows that vertex 1 is dominating in T_2, i.e., it is adjacent to vertex j, for $j = 2, \ldots, n$. If vertex 1 is also dominating in T_1, then we can complete the proof by induction by passing to the threshold graphs obtained from T_1, T_2 by dropping vertex 1. If vertex 1 is not dominating in T_1, then by Lemma 2.7(ii), there is an $e \in \binom{[n]}{2} - E_1$ containing the vertex 1 such that $T_3 = ([n], E_1 \cup \{e\})$ is threshold. Note that $|(E_1 \cup \{e\}) \oplus E_2| = |E_1 \oplus E_2| - 1$. The proof is now completed as in case (a). $\square$

The rest of this section is devoted to giving a negative answer to a question of Golumbic [11]. Let $<_L$ be a linear order on $[n]$. For $x, y \in [n]$ we write $x \leqslant_L y$ if $x = y$ or $x <_L y$. A *finite multiset* is a set with repeated elements such that the total number of elements (counted with repetitions) is finite. This finite number is called the *cardinality* of the finite multiset. The union of two multisets is defined in the obvious way. For instance, the union of the multisets $\{1, 1, 2, 3, 3\}$ and $\{1, 2, 4, 4, 3\}$ is $\{1, 1, 1, 2, 2, 3, 3, 3, 4, 4\}$. A finite multiset on $[n]$ is a finite multiset set all of whose elements belong to $[n]$. Let $M(n, k)$ denote the set of all finite multisets on $[n]$ with cardinality k. We define a partial order on $M(n, k)$, with respect to the linear order $<_L$ on $[n]$, and also denoted $\leqslant_L$, as follows: Given $X, Y \in M(n, k)$ write $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$, where $x_1 \leqslant_L x_2 \leqslant_L \cdots \leqslant_L x_k$ and $y_1 \leqslant_L y_2 \leqslant_L \cdots \leqslant_L y_k$. Then $X \leqslant_L Y$ provided $x_i \leqslant_L y_i$ for all i.

Definition 2.11. Let $<_L$ be a linear order on $[n]$. Let q be a positive integer. A subset $I \subseteq \binom{[n]}{r}$ is a *q-ideal*, w.r.t $<_L$, provided the following condition holds: Let $X_1, X_2, \dots, X_q$ be any q elements of I and let $Y_1, Y_2, \dots, Y_q$ be any q elements of $\binom{[n]}{r} - I$. Form the multiset unions $X = X_1 \cup X_2 \cup \cdots \cup X_q \in M(n, rq)$ and $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_q \in M(n, rq)$. Then $Y \nleqslant_L X$.

A 1-ideal is also called an *order ideal*.

Theorem 2.12. *Let $H = ([n], E)$ be an r-threshold hypergraph. Assume that H is determined by the weights $(c_1, \dots, c_n)$ and let π be a permutation of $[n]$ satisfying $c_{\pi(1)} \geqslant c_{\pi(2)} \geqslant \cdots \geqslant c_{\pi(n)}$. Define a linear order $<_L$ on $[n]$ by $\pi(1) <_L \pi(2) <_L \cdots <_L \pi(n)$. Then, for all positive integers q, E is a q-ideal of $\binom{[n]}{r}$ w.r.t $<_L$.*

Proof. Let $X_1, X_2, \dots, X_q \in E$ and $Y_1, Y_2, \dots, Y_q \in \binom{[n]}{r} - E$. Write the multiset union $X_1 \cup X_2 \cup \cdots \cup X_q$ as $\{a_1, a_2, \dots, a_{rq}\}$ where $a_1 \leqslant_L a_2 \leqslant_L \cdots \leqslant_L a_{rq}$ and the multiset union $Y_1 \cup Y_2 \cup \cdots \cup Y_q$ as $\{b_1, b_2, \dots, b_{rq}\}$, where $b_1 \leqslant_L b_2 \leqslant_L \cdots \leqslant_L b_{rq}$. Since $X_j \in E$ for all j, we have $\sum_{i=1}^{rq} c_{a_i} \geqslant 0$. Since $Y_j \notin E$ for all j, we have $\sum_{i=1}^{rq} c_{b_i} < 0$.

If $Y \leqslant_L X$, then $b_i \leqslant_L a_i$ and thus $c_{b_i} \geqslant c_{a_i}$ for $i = 1, 2, \dots, rq$. It follows that $\sum_{i=1}^{rq} c_{b_i} \geqslant 0$, a contradiction. Thus E is a q-ideal. □

The following result was proved by Chvátal and Hammer [5]. In order to keep this paper self-contained, we include a short proof.

Theorem 2.13. *Let $T = ([n], E)$ be a graph. Then T is threshold if and only if there is a linear order $<_L$ on $[n]$ such that E is an order ideal w.r.t $<_L$.*

Proof. (*Only if*) This is a special case of Theorem 2.12.

(*If*) Without loss of generality we may assume that $<_L$ is the usual linear order, i.e., $1 <_L 2 <_L \cdots <_L n$. We may now write $<$ for $<_L$. An element $e \in E$ is maximal if $e \leqslant f$, $f \in E$ implies $f = e$. We shall write elements of $\binom{[n]}{2}$ as $\{i, j\}$, with $i < j$. Let S denote the set of maximal elements of E. Write S as $\{\{i_1, j_t\},$

$\{i_2, j_{t-1}\}, \ldots, \{i_t, j_1\}\}$, where $i_1 < i_2 < \cdots < i_t$. It follows that $j_1 < j_2 < \cdots < j_t$. We thus have $i_1 < i_2 < \cdots < i_t < j_1 < j_2 < \cdots < j_t$. Put $i_0 = 0$ and $j_0 = i_t$. Define sets $A_1, \ldots, A_t$ and $B_1, B_2, \ldots, B_{t+1}$ by $A_k = \{i_{k-1}+1, \ldots, i_k\}$, $B_k = \{j_{k-1}+1, \ldots, j_k\}$, $k = 1, \ldots, t$ and $B_{t+1} = \{j_t + 1, \ldots, n\}$. Note that $[n] = A_1 \cup \cdots \cup A_t \cup B_1 \cup \cdots \cup B_t \cup B_{t+1}$ (disjoint union). Define weights $(c_1, c_2, \ldots, c_n)$ as follows: elements of A_l get weight $t - l + 1$ and elements of B_l get weight $-l$. We claim that $([n], E)$ is a threshold graph determined by the weights $(c_1, c_2, \ldots, c_n)$. This follows from the following three observations:

(a) We have $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n$.
(b) Let $\{i_l, j_{t-l+1}\} \in S$. Then $i_l \in A_l$ and $j_{t-l+1} \in B_{t-l+1}$. Thus $c_{i_l} + c_{j_{t-l+1}} = 0$.
(c) Let $\{i_l, j_{t-l+1}\} \in S$. Then $c_{i_l+1} + c_{j_{t-l+1}} < 0$ and $c_{i_l} + c_{j_{t-l+1}+1} < 0$. $\square$

Golumbic [11] asked whether Theorem 2.13 is true for r-hypergraphs. We now answer this question in the negative. We first need an elementary lemma.

Lemma 2.14. *Let $H = ([n], E)$ be an r-threshold hypergraph. Let $<_L$ be a linear order on $[n]$ such that E is an order ideal of $\binom{[n]}{r}$ w.r.t $<_L$. Let π be the unique permutation of $[n]$ satisfying $\pi(1) <_L \pi(2) <_L \cdots <_L \pi(n)$. Then the weights $(c_1, c_2, \ldots, c_n)$ determining H can be chosen so that $c_{\pi(1)} \geqslant c_{\pi(2)} \geqslant \cdots \geqslant c_{\pi(n)}$.*

Proof. Without loss of generality we can assume that π is the identity. Let $c = (c_1, c_2, \ldots, c_n)$ be weights determining H. Assume that $c_1 < c_i$ for some i. Switch c_1 and c_i in $(c_1, c_2, \ldots, c_n)$ to obtain $d = (d_1, d_2, \ldots, d_n)$. We claim that $d = (d_1, d_2, \ldots, d_n)$ also determines H. Let $Y \in \binom{[n]}{r}$. If $\{1, i\} \subseteq Y$ or $\{1, i\} \cap Y = \emptyset$, then $d(Y) = c(Y)$ and there is nothing to check. If $|\{1, i\} \cap Y| = 1$, consider the following cases:

(a) $1 \in Y,\ i \notin Y$: If $c(Y) \geqslant 0$, then clearly $d(Y) \geqslant 0$. If $c(Y) < 0$, then $Y \notin E$ and, since E is an order ideal w.r.t identity, we have $Y - \{1\} \cup \{i\} \notin E$ and thus $d(Y) < 0$.
(b) $1 \notin Y, i \in Y$: Similar to case (a).

Thus $d = (d_1, d_2, \ldots, d_n)$ also determines H. By repeating this process we can get weights satisfying the desired condition. $\square$

Example 2.15. Consider the set [9] with the usual linear order, i.e., $1 <_L 2 <_L \cdots <_L 9$. Define an order ideal E of $\binom{[9]}{3}$ w.r.t $<_L$ as follows: $E = \{X \in \binom{[9]}{3}$: $X \leqslant_L \{1, 5, 9\}$ or $X \leqslant_L \{2, 6, 8\}$ or $X \leqslant_L \{3, 4, 7\}\}$. Clearly $X_1 = \{1, 5, 9\}$, $X_2 = \{2, 6, 8\}$, $X_3 = \{3, 4, 7\} \in E$.

We can check that $Y_1 = \{2, 3, 9\}$, $Y_2 = \{3, 5, 6\}$, $Y_3 = \{1, 7, 8\} \notin E$. Now consider the multiset unions

$$X = X_1 \cup X_2 \cup X_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \in M(9, 9).$$

$$Y = Y_1 \cup Y_2 \cup Y_3 = \{1, 2, 3, 3, 5, 6, 7, 8, 9\} \in M(9, 9).$$

We have $Y \leqslant_L X$ and thus E is not a 3-ideal w.r.t $<_L$. Assume that $H = ([9], E)$ were 3-threshold. By Lemma 2.14 we can choose weights $(c_1, c_2, \ldots, c_9)$ determining H so that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_9$. It follows from Theorem 2.12 that E would be a 3-ideal w.r.t $<_L$, a contradiction. Thus, H is not 3-threshold.

3. Facets

In this section we study the facets of the polytope $D_n(r)$. Some results on ranks of incidence matrices used in this section are collected in Section 4.

Theorem 3.1. *Let n, r be positive integers with $n \geqslant r + 1$. Then the polytope $D_n(r)$ is full dimensional.*

Proof. Let M denote the $\binom{n}{r} \times n$ matrix, with rows indexed by elements $A \in \binom{[n]}{r}$, columns indexed by elements $i \in [n]$, and with $M(A, i) = 1$ if $i \in A$ and $M(A, i) = 0$ if $i \notin A$. Note that every row of M is the degree sequence of an r-hypergraph on $[n]$ with a single edge. By Theorem 4.3, the rank of M is n. The degree sequence $(0, 0, \ldots, 0)$, together with n linearly independent rows of M, forms an affinely independent set of $n + 1$ degree sequences. It follows that the dimension of $D_n(r)$ is n. $\square$

Definition 3.2. Let n, r be positive integers with $n \geqslant r$. Given an n-tuple $a = (a_1, \ldots, a_n)$ of integers, define

$$F_r(a) = \sum_X a(X),$$

where the sum is over all $X \in \binom{[n]}{r}$ satisfying $a(X) > 0$.

By convention, the empty sum is taken to be zero. Note that permuting $(a_1, a_2, \ldots, a_n)$ does not change the value of $F_r(a_1, a_2, \ldots, a_n)$. Note also that in the definition we could have taken the sum over all $X \in \binom{[n]}{r}$ satisfying $a(X) \geqslant 0$.

Example 3.3.

(i) $F_3(2, -1, 0, 2, -2) = (2 - 1 + 0) + (2 - 1 + 2) + (2 + 0 + 2) + (2 + 2 - 2) + (-1 + 0 + 2) = 11.$

(ii) Consider sets $S, T \subseteq [n]$ $(n \geqslant r)$ with $S \cap T = \emptyset$. Define $a = (a_1, a_2, \ldots, a_n)$ by

$$a_i = \begin{cases} 1 & \text{if } i \in S, \\ -(r-1) & \text{if } i \in T, \\ 0 & \text{if } i \notin S \cup T. \end{cases}$$

Now, for $X \in \binom{[n]}{r}$, we have $a(X) > 0$ if and only if $X \cap T = \emptyset$ and $X \cap S \neq \emptyset$. Let N denote the 0–1 matrix with columns indexed by $[n] - T$, with rows indexed by the set $\{X \in \binom{[n]}{r}\colon\ X \cap T = \emptyset,\ \ X \cap S \neq \emptyset\}$, and with entry $N(X, i) = 1$ if $i \in X \cap S$ and equal to 0 otherwise. It is easily seen that $F_r(a)$ is the sum of all the elements of N. For $i \in ([n] - T) - S$, the column of N corresponding to i will have all entries equal to 0. For $i \in ([n] - T) \cap S$, the number of 1's in the column of N corresponding to i is $\binom{n-1-|T|}{r-1}$. Thus $F_r(a) = |S|\binom{n-1-|T|}{r-1}$.

Note that, when $r = 2$, we have $F_2(a) = |S|(n - 1 - |T|)$.

(iii) Consider sets $S, T \subseteq [n] (n \geqslant r)$ with $S \cap T = \emptyset$. Define $b = (b_1, b_2, \ldots, b_n)$ by

$$b_i = \begin{cases} r-1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{if } i \notin S \cup T. \end{cases}$$

Let N denote the matrix with entries in $\{0, -1, r-1\}$, with columns indexed by $[n]$, with rows indexed by the set

$$\left\{X \in \binom{[n]}{r}\colon\ b(X) \geqslant 0\right\},$$

and with entry

$$N(X, i) = \begin{cases} r-1 & \text{if } i \in X \cap S, \\ -1 & \text{if } i \in X \cap T, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $F_r(b)$ is the sum of the elements of N. If $i \in S$, the column of N corresponding to i has $\binom{n-1}{r-1}$ nonzero entries, all equal to $(r-1)$. If $i \in T$ then, for $X \in \binom{[n]}{r}$ with $i \in X$, we have $b(X) \geqslant 0$ if and only if $X \cap S \neq \emptyset$. Thus the column of N corresponding to i has $\binom{n-1}{r-1} - \binom{n-1-|S|}{r-1}$ nonzero entries, all equal to -1. If $i \in [n] - (S \cup T)$, the column of N corresponding to i has no nonzero entries. Thus

$$F_r(b) = |S|(r-1)\binom{n-1}{r-1} + |T|\left(\binom{n-1-|S|}{r-1} - \binom{n-1}{r-1}\right).$$

Note that, when $r = 2$, we have $F_2(b) = |S|(n - 1 - |T|)$.

Explicit formulae for $F_r(a_1, a_2, \ldots, a_n)$ exist only for special values of $(a_1, a_2, \ldots, a_n)$. But the main point is that, for fixed r, given $a = (a_1, a_2, \ldots, a_n)$ we can calculate $F_r(a)$ in polynomial time.

Lemma 3.4. *Let $a = (a_1, a_2, \ldots, a_n)$ be an integral n-tuple and let b be a real number. The inequality*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leqslant b \tag{5}$$

is valid for $D_n(r)$ if and only if $F_r(a) \leqslant b$.

Proof. (*If*) Let $H = ([n], E)$ be an r-hypergraph with degree sequence $(d_1, d_2, \ldots, d_n)$. Then, as in the proof of Theorem 2.3, we have $a_1d_1 + a_2d_2 + \cdots + a_nd_n = \sum a(X)$, where the sum is over all edges $X \in E$. This sum is clearly $\leqslant F_r(a)$. It follows that inequality (5) is valid for $D_n(r)$.

(*Only if*) Define $H = ([n], E)$ by setting E equal to the collection of all r-subsets X of $[n]$ satisfying $a(X) > 0$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of H. Then $a_1d_1 + a_2d_2 + \cdots + a_nd_n = F_r(a)$. Since (5) is valid, we have $F_r(a) \leqslant b$. $\square$

Corollary 3.5. *Let p, q be positive integers. The inequality*

$$\sum_{i\in S} px_i - \sum_{i\in T} qx_i \leqslant F_r(p, \ldots, p, -q, \ldots, -q, 0, \ldots, 0) \tag{6}$$

is valid for $D_n(r)$, for all sets $S, T \subseteq [n]$, $S \cap T = \emptyset$.

(Here, in the expression on the r.h.s, p is repeated $|S|$ times, $-q$ is repeated $|T|$ times, and 0 is repeated $n - |S| - |T|$ times.)

When $r = 2$ and $p = q = 1$, it follows from Example 3.3(ii) that inequality (6) reduces to the Erdös–Gallai inequality ((4) of Section 1). We shall prove later that, in many cases, (6) determines a facet of $D_n(r)$.

Definition 3.6. Let n, r be positive integers with $n \geqslant r$. Let $a = (a_1, a_2, \ldots, a_n)$ be an integral n-tuple. Define $M_r(a)$ to be the n column matrix whose rows are the characteristic vectors of r-subsets X of $[n]$ satisfying $a(X) = 0$.

Example 3.7. Let $n = 5$, $r = 3$, and $a = (1, 1, 1, -2, -2)$. Then

$$M_3(a) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Since the extreme points of $D_n(r)$ have rational (infact, integral) coordinates, the linear inequalities determining the facets of $D_n(r)$ can be chosen to have integer coefficients.

Theorem 3.8. *Let $a = (a_1, a_2, \ldots, a_n)$ be an integral n-tuple and let b be a real number. Let $n \geqslant r + 1$. The linear inequality*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leqslant b \tag{7}$$

is a facet of $D_n(r)$ if and only if $b = F_r(a)$ and $\operatorname{rank}(M_r(a)) = n - 1$.

Proof. (*If*) Let A be the set of all r-subsets X of $[n]$ satisfying $a(X) > 0$ and let B be the set of all r-subsets X of $[n]$ satisfying $a(X) = 0$. The degree sequence d_0 of the hypergraph $H_0 = ([n], A)$ satisfies (7) with equality. Moreover, by Remark 2.4, the hypergraphs whose degree sequences satisfy (7) with equality are of the form $([n], A \coprod C)$, where $C \subseteq B$. Let $B = \{X_1, X_2, \ldots, X_t\}$. Consider the hypergraphs $H_1 = ([n], A \cup \{X_1\}),\ H_2 = ([n], A \cup \{X_2\}), \ldots, H_t = ([n], A \cup \{X_t\})$, with degree sequences $d_1, d_2, \ldots, d_t$, respectively, all satisfying (7) with equality. The $t \times n$ matrix, with ith row equal to $d_i - d_0$, is precisely $M_r(a)$. Since rank $M_r(a) = n - 1$, it follows that we have n affinely independent degree sequences satisfying (7) with equality. Thus (7) determines a facet.

(*Only if*) By Lemma 3.4 we have $F_r(a) \leqslant b$. If $F_r(a) < b$, then no degree sequence can satisfy (7) with equality. So we must have $b = F_r(a)$. Let $A, B, X_1, \ldots, X_t, d_0, \ldots, d_t$ be defined as in the if part. The degree sequences satisfying (7) with equality are $\{d_0 + \sum_{i \in S}(d_i - d_0)\colon\ S \subseteq [t]\}$. The affine hull of this set is the same as the affine hull of the set $\{d_0, d_1, \ldots, d_t\}$. Since (7) is a facet the affine rank of the set $\{d_0, d_1, \ldots, d_t\}$ is n or, equivalently, the rank (or linear rank) of the set $\{d_1 - d_0, d_2 - d_0, \ldots, d_t - d_0\}$ is $n - 1$. Since the rows of $M_r(a)$ are precisely $d_i - d_0,\ i = 1, 2, \ldots, t$, we have rank $M_r(a) = n - 1$. $\square$

Example 3.9. Let n, r and a be as in Example 3.7. It can be checked that rank $M_3(a) = 4$. Since $F_3(a) = 3$, we have from Theorem 3.8 that $x_1 + x_2 + x_3 - 2x_4 - 2x_5 \leqslant 3$ is a facet of $D_5(3)$.

We now give a bound on the sizes of the coefficients appearing in the facet determining inequalities of $D_n(r)$.

Theorem 3.10. *Let $n \geqslant r + 1$ and let $a = (a_1, a_2, \ldots, a_n)$ be an integral n-tuple with g.c.d $(a_1, a_2, \ldots, a_n) = 1$ and such that the linear inequality*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leqslant F_r(a) \tag{8}$$

determines a facet of $D_n(r)$. Then $|a_i| \leqslant nr^{n/2},\ i = 1, \ldots, n$ and

$$F_r(a) \leqslant n\binom{n}{r}r^{(n+2)/2}.$$

Proof. It is enough to prove the bound on $|a_i|$, as then the bound on $F_r(a)$ follows immediately.

Let N be the $(n - 1) \times n$ matrix consisting of some $n - 1$ linearly independent rows of $M_r(a)$. The column vector a^{t} is a solution of the homogeneous system of equations $Nx = 0$ (where $x = (x_1, \ldots, x_n)^{\mathrm{t}}$ is the column vector of unknowns). Since rank(N) = $n - 1$, all integral solutions of $Nx = 0$ are rational multiples of a^t.

Without loss of generality we may assume that the first $n - 1$ columns of N are linearly independent. Let R denote the submatrix of N formed by the first $n - 1$

columns and let $c = (c_1, \dots, c_{n-1})^{\mathrm{t}}$ denote the last column of N. Every solution of $Nx = 0$ is of the form $x_n = \alpha$, and $(x_1, \dots, x_{n-1})^{\mathrm{t}} = -R^{-1}(c_1\alpha, \dots, c_{n-1}\alpha)^{\mathrm{t}}$, for some arbitrary α.

By Cramer's rule the vector $(y_1, y_2, \dots, y_n)^{\mathrm{t}}$ given by $y_n = \det R$ and $(y_1, \dots, y_{n-1})^{\mathrm{t}} = (-\det R)R^{-1}(c_1, \dots, c_{n-1})^{\mathrm{t}}$ is an integral solution to $Nx = 0$. Since the length of each row of R is atmost $r^{1/2}$, it follows from Hadamard's inequality that $y_n = \det R \leqslant r^{(n-1)/2}$. Similarly, the absolute values of the entries of the adjoint of R are $\leqslant r^{(n-2)/2}$. Since each c_i is 0 or 1, by Cramer's rule it follows that $|y_i| \leqslant (n-1)r^{(n-2)/2}$, $i = 1, \dots, n-1$. Thus $\max\{|y_1|, \dots, |y_n|\} \leqslant nr^{n/2}$.

Let $d =$ g.c.d $(y_1, \dots, y_n)$ and $y' = (1/d)(y_1, \dots, y_n)$. We claim that $a = y'$ or $a = -y'$. This will prove the theorem. Write $(a_1, \dots, a_n) = (\alpha/d)(y_1, \dots, y_n)$, where $\alpha = p/q$ is rational. Then $q(a_1, \dots, a_n) = p(y_1/d, \dots, y_n/d)$. Taking g.c.d on both sides we see that $|p| = |q|$. $\square$

It follows from the result above that the bit length of the numbers appearing in the facet determining inequalities is $\mathrm{O}(n)$. In Theorem 3.14, we shall produce an explicit facet where this bound is achieved. Using Theorem 3.8 we now determine the facets of $D_n(2)$.

Theorem 3.11. *The facets of $D_n(2)$, $n \geqslant 3$, are given by*

$$\sum_{i \in S} x_i - \sum_{i \in T} x_i \leqslant |S|(n - 1 - |T|), \tag{9}$$

for all sets $S, T \subseteq [n]$, $S \cap T = \emptyset$ satisfying one of the following conditions:

(i) $|S \cup T| = 1$, $|\overline{S \cup T}| \geqslant 3$.
(ii) $|S| \geqslant 1$, $|T| \geqslant 1$, $|\overline{S \cup T}| = 0$ *or* $|\overline{S \cup T}| \geqslant 3$.

Proof. Let $n \geqslant 3$ and let

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leqslant b \tag{10}$$

define a facet of $D_n(2)$, where $a = (a_1, a_2, \dots, a_n)$ is an integral n-tuple and b is an integer. Put $S = \{i \in [n]\colon a_i > 0\}$ and $T = \{i \in [n]\colon a_i < 0\}$. If $S \cup T = \emptyset$, then the solution set of (10) is either empty or all of $\mathbb{R}^n$, and thus cannot determine a facet of $D_n(2)$. So we may assume that $S \cup T \neq \emptyset$. After multiplying, if necessary, by a positive real number we may also assume that g.c.d $\{a_i\colon i \in S \cup T\} = 1$. We shall show that $|a_i| = 1$, $i \in S \cup T$ and that S, T satisfy (precisely) one of the conditions (i) or (ii) stated above. It will then follow from Theorem 3.8 and Example 3.3 (ii) that $b = F_2(a) = |S|(n - 1 - |T|)$. This will complete the proof of the theorem.

If a 2-subset $\{i_1, i_2\} \subseteq [n]$ satisfies $a_{i_1} + a_{i_2} = 0$, then either $\{i_1, i_2\} \subseteq \overline{S \cup T}$ or $|\{i_1, i_2\} \cap S| = |\{i_1, i_2\} \cap T| = 1$. Put $E_1 = \{\{i_1, i_2\} \subseteq S \cup T\colon\ a_{i_1} + a_{i_2} = 0\}$ and $E_2 = \binom{\overline{S \cup T}}{2}$. Let N_1 be the matrix whose columns are indexed by $S \cup T$ and whose rows are characteristic vectors of E_1. If S or T is empty, we take N_1 to be the empty matrix with rank zero. Let N_2 be the matrix whose columns are indexed by $\overline{S \cup T}$ and whose rows are characteristic vectors of E_2. Thus $M_2(a)$ has the block form:

$$\begin{array}{c c c c} & S \quad T & \overline{S \cup T} & \\ E_1 & \left[\begin{array}{c|c} N_1 & O \\ \hline O & N_2 \end{array}\right. & \left.\right] & E_2 \end{array}$$

We have $n - 1 = \mathrm{rank}(M_2(a)) = \mathrm{rank}(N_1) + \mathrm{rank}(N_2)$. Note that N_1 is the edge-vertex incidence matrix of the bipartite graph $B = (S, T; E_1)$. We now consider the following two cases:

(i) $S \neq \emptyset$ *and* $T \neq \emptyset$: By Theorem 4.1, $\mathrm{rank}(N_1) \leqslant |S| + |T| - 1$. Since rank $(M_2(a)) = n - 1$, we must have $\mathrm{rank}(N_1) = |S| + |T| - 1$ and $\mathrm{rank}(N_2) = |\overline{S \cup T}|$. By Theorems 4.1 and 4.3 this happens if and only if B is connected and $|\overline{S \cup T}| = 0$ or $\geqslant 3$. Since B is connected we have $|a_i| = |a_j|,\ i, j \in S \cup T$. Since g.c.d $\{a_i\colon\ i \in S \cup T\} = 1$ we have $|a_i| = 1, i \in S \cup T$.

(ii) *At least one of S or T, but not both, is empty*: In this case we have $\mathrm{rank}(M_2(a)) = n - 1$ if and only if $\mathrm{rank}(N_2) = n - 1$. Since $|\overline{S \cup T}| \leqslant n - 1$, it follows from Theorem 4.3 that $\mathrm{rank}(N_2) = n - 1$ iff $|\overline{S \cup T}| = n - 1$ and $|\overline{S \cup T}| \geqslant 3$. Since $|S \cup T| = 1$ and g.c.d $\{a_i\colon\ i \in S \cup T\} = 1$ we have $|a_i| = 1,\ i \in S \cup T$. □

We now give some Erdös–Gallai type facets of $D_n(r)$. The notation on the r.h.s of (11) below is as in (6).

Theorem 3.12. *Let n, p, q, r be positive integers with $n \geqslant r + 1$. Let $S, T \subseteq [n]$ with $S \cap T = \emptyset$. The inequality*

$$\sum_{i \in S} p x_i - \sum_{i \in T} q x_i \leqslant F_r(p, \ldots, p, -q, \ldots, -q, 0, \ldots, 0) \tag{11}$$

defines a facet of $D_n(r)$ whenever any of the following conditions holds:

(i) $p = q = 1,\ |S \cup T| = 1,\ |\overline{S \cup T}| \geqslant r + 1$.
(ii) $p + q = r,\ |\overline{S \cup T}| = 0,\ p = 1,\ |T| = 1,\ |S| = q + 1$.
(iii) $p + q = r,\ |\overline{S \cup T}| = 0,\ q = 1,\ |S| = 1,\ |T| = p + 1$.
(iv) $p + q = r,\ |\overline{S \cup T}| = 0,\ |T| \geqslant p + 1,\ |S| \geqslant q + 1$.
(v) $p + q \leqslant r,\ |\overline{S \cup T}| \geqslant r + 1,\ p = q = 1,\ |S| = |T| = 1$.
(vi) $p + q \leqslant r,\ |\overline{S \cup T}| \geqslant r + 1,\ p = 1,\ |T| = 1,\ |S| = q + 1$.
(vii) $p + q \leqslant r,\ |\overline{S \cup T}| \geqslant r + 1,\ q = 1,\ |S| = 1,\ |T| = p + 1$.
(viii) $p + q \leqslant r,\ |\overline{S \cup T}| \geqslant r + 1,\ |T| \geqslant p + 1,\ |S| \geqslant q + 1$.

Proof. Let $a = (a_1, a_2, \ldots, a_n)$ be defined by $a_i = p$ for $i \in S$, $a_i = -q$ for $i \in T$, and $a_i = 0$ for $i \in \overline{S \cup T}$. We have to check that $\operatorname{rank}(M_r(a)) = n - 1$ in each of the cases (i)–(viii) above.

(i) Assume, without loss of generality, that $S = \{n\}$ and $T = \emptyset$. Then $M_r(a)$ is the $\binom{n-1}{r} \times n$ matrix, whose rows are characteristic vectors of r-subsets of $[n]$ not containing n. Since $(n-1) \geqslant r + 1$, it follows from Theorem 4.3 that $\operatorname{rank}(M_r(a)) = n - 1$.

(ii), (iii) and (iv) Let $E = \{X \subseteq [n]\colon\ |X \cap S| = q,\ |X \cap T| = p\}$. $M_r(a)$ is the matrix with columns indexed by $S \cup T$ and rows indexed by characteristic vectors of elements of E. It follows from Theorem 4.5 that $\operatorname{rank}(M_r(a)) = n - 1 = |S| + |T| - 1$ whenever $p, q, |S|, |T|$ satisfy the conditions in cases (ii)–(iv) above.

(v), (vi), (vii) and (viii) Let N_2 be the matrix with columns indexed by elements of $\overline{S \cup T}$ and rows indexed by characteristic vectors of elements of $E_2 = \binom{\overline{S \cup T}}{r}$.

Let $E_1 = \{X \subseteq S \cup T\colon |X \cap S| = q, |X \cap T| = p\}$. Let N_1 be the matrix with columns indexed by $S \cup T$ and rows indexed by characteristic vectors of elements of E_1.

Fix a subset $A \subseteq \overline{S \cup T},\ |A| = r - (p + q)$. Let N_3 be the matrix with columns indexed by $\overline{S \cup T}$, and with $|E_1|$ rows, where each row is the characteristic vector of A.

Consider the matrix N, shown in block form below.

$$
\begin{array}{c c c}
 & \begin{array}{cc} S \quad T & \overline{S \cup T} \end{array} & \\
E_1 & \left[\begin{array}{c|c} N_1 & N_3 \\ \hline O & N_2 \end{array}\right] & \\
 & & E_2
\end{array}
$$

By Theorem 4.3, $\operatorname{rank}(N_2) = |\overline{S \cup T}|$. By Theorem 4.5, $\operatorname{rank}(N_1) = |S| + |T| - 1$ whenever $p, q, |S|, |T|$ satisfy the conditions of cases (v)–(viii). Thus $\operatorname{rank}(N) \geqslant n - 1$. Since the rows of N are contained in the rows of $M_r(a)$, we have $\operatorname{rank}(M_r(a)) \geqslant n - 1$. In fact $\operatorname{rank}(M_r(a)) = n - 1$, since every row of $M_r(a)$ is orthogonal to the nonzero vector a. $\square$

We now describe a simple lifting procedure, whereby facets of $D_n(r)$ can be lifted to produce facets of $D_{n+1}(r)$. This procedure will then be used to produce a facet of $D_n(3)$ with Fibonacci numbers as coefficients.

Lemma 3.13. *Let n, r be positive integers with $n \geqslant r + 1$. Let $a = (a_1, a_2, \ldots, a_n)$ be an integral n-tuple such that the inequality*

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leqslant F_r(a)$$

is facet inducing for $D_n(r)$. Fix $i \in [n+1]$. Choose a subset $X \subseteq [n]$ of cardinality $(r-1)$ and put $d = -a(X)$ and $b = (a_1, a_2, \dots, a_n, d)$. Then the inequality

$$a_1x_1 + a_2x_2 + \cdots + a_{i-1}x_{i-1} + dx_i + a_ix_{i+1} + \cdots + a_nx_{n+1} \leqslant F_r(b)$$

is facet inducing for $D_{n+1}(r)$.

Proof. Without loss of generality we may assume that $i = n+1$. Let v_X be the characteristic vector of X, as a subset of $[n]$. The rows of the following matrix, depicted in block form:

$$\begin{array}{c} \begin{array}{cc} 1 \quad 2 \quad \dots \quad n & \quad n+1 \end{array} \\ \left[\begin{array}{c|c} & 0 \\ M_{\mathrm{r}}(a) & \vdots \\ & 0 \\ \hline & \\ v_X & 1 \\ & \end{array}\right] \end{array}$$

certainly belong to $M_r(b)$. Since $\operatorname{rank}(M_r(a)) = n-1$, it follows that $\operatorname{rank}(M_r(b)) \geqslant n$. Since every row of $M_r(b)$ is orthogonal to the nonzero vector b, we have rank $(M_r(b)) = n$. This completes the proof. $\square$

Define the *Fibonacci* numbers $f_0, f_1, f_2, f_3, \dots$ by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, for $n \geqslant 2$. It is well known that $f_n = (1/\sqrt{5})(r_+^n - r_-^n)$, $n \geqslant 0$, where $r_+ = 1 + \sqrt{5}/2$ and $r_- = 1 - \sqrt{5}/2$. In fact, it can be shown that f_n is the integer nearest to $r_+^n/\sqrt{5}$ (see [17]). In the following result, we shall be considering the polytope $D_{2n+3}(3)$. For convenience, we shall relabel the $2n+3$ variables $\{x_1, x_2, \dots, x_{2n+3}\}$ as $\{x_i, y_i\colon\ i = 1, 2, \dots, n\} \cup \{z_1, z_2, z_3\}$.

Theorem 3.14. *For $n \geqslant 1$, the inequality*

$$\sum_{i=1}^{n} f_ix_i - \sum_{i=1}^{n} f_iy_i \leqslant F_3(f_1, \dots, f_n, -f_1, \dots, -f_n, 0, 0, 0) \tag{12}$$

is facet inducing for $D_{2n+3}(3)$.

Proof. By induction on n. For $n = 1, 2$, inequality (12) is

$$x_1 - y_1 + 0z_1 + 0z_2 + 0z_3 \leqslant F_3(1, -1, 0, 0, 0) \tag{13}$$

$$x_1 + x_2 - y_1 - y_2 + 0z_1 + 0z_2 + 0z_3 \leqslant F_3(1, 1, -1, -1, 0, 0, 0) \tag{14}$$

Let $a = (1, -1, 0, 0, 0)$. Then

$$M_3(a) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

It can be checked that $\mathrm{rank}(M_3(a)) = 4$. Then (13) is facet inducing for $D_5(3)$. Applying Lemma 3.13 twice to (13) we see that (14) is facet inducing for $D_7(3)$. Now assume that (12) is facet inducing for $D_{2n+3}(3)$ for some $n \geqslant 2$. Again applying Lemma 3.13 twice we see that

$$\begin{aligned} &\sum_{i=1}^{n} f_i x_i + (f_n + f_{n-1})x_{n+1} - \sum_{i=1}^{n} f_i y_i - (f_n + f_{n-1})y_{n+1} \\ &\qquad \leqslant F_3(f_1, \ldots, f_n, f_n + f_{n-1}, -f_1, \ldots, -f_n, -f_n - f_{n-1}, 0, 0, 0) \end{aligned}$$

is facet inducing for $D_{2n+5}(3)$. Since $f_{n+1} = f_n + f_{n-1}$, the proof is complete. □

Clearly, we can get more facets from (12) by permuting the l.h.s.

4. Incidence matrices

In this section we consider some classes of incidence matrices. Results on the ranks of these matrices were used in Section 3.

The following two theorems are well known. See, for instance, [17].

Theorem 4.1. *Let $B = (S, T; E)$, where S and T are disjoint, be a bipartite graph with vertex set $S \cup T$ and edge set E. Let M_B denote the edge-vertex incidence matrix of B, i.e., the $|E| \times (|S| + |T|)$ matrix, with rows indexed by E and columns indexed by $S \cup T$, and with $M_B(e, v)$, $e \in E, v \in S \cup T$, equal to 1 if e is incident with v and equal to 0 if e is not incident with v. Then* $\mathrm{rank}(M_B) = |S| + |T| -$ (*number of components of B*).

Let i, j, n be nonnegative integers with $i, j \leqslant n$. Let $I_n(j, i)$ denote the $\binom{n}{j} \times \binom{n}{i}$ matrix, with rows indexed by elements of $\binom{[n]}{j}$, with columns indexed by elements of $\binom{[n]}{i}$, and with entry in row X, column Y equal to 1 if $X \supseteq Y$ and equal to 0 if $X \not\supseteq Y$.

Theorem 4.2. *The columns of $I_n(j, i)$ are linearly independent if and only if $i = j$ or $i < j \leqslant n - i$.*

We shall only need the following special case of Theorem 4.2.

Theorem 4.3. *Let n, r be positive integers with $r \leqslant n$. Then the $\binom{n}{r} \times n$ matrix, whose rows are the characteristic vectors of r-subsets of $[n]$, has full column rank if and only if $n = 1$ or $n \geqslant r + 1$.*

Let i, j, n, k, l, m be nonnegative integers satisfying $i, j \leqslant n,\ k, l \leqslant m (i = j$ or $i < j \leqslant n - i)$, and $(k = l$ or $k < l \leqslant m - k)$. We define a matrix $N_{n,m}(j, l, i, k)$ with $\binom{n}{j}\binom{m}{l}$ rows and $\binom{n}{i} + \binom{m}{k}$ columns as follows: The columns are partitioned into two sets, with one set indexed by the elements of $\binom{[n]}{i}$ and the other set indexed by the elements of $\binom{[m]}{k}$. The rows are indexed by the elements of $\binom{[n]}{j} \times \binom{[m]}{l}$. For $(A, B) \in \binom{[n]}{j} \times \binom{[m]}{l}$, $C \in \binom{[n]}{i}$, $D \in \binom{[m]}{k}$, the entry in row (A, B), column C is 1 if $A \supseteq C$ and 0 if $A \not\supseteq C$ and the entry in row (A, B), column D is 1 if $B \supseteq D$ and 0 if $B \not\supseteq D$. Though we do not use the full generality of the following result, it may be useful in other situations involving incidence matrices.

Theorem 4.4. *The rank of $N_{n,m}(j, l, i, k)$ is $\binom{n}{i} + \binom{m}{k} - 1$.*

Proof. We write N for $N_{n,m}(j, l, i, k)$ and N_2 for $N_{n,m}(j, k, i, k)$. Define an $\binom{n}{j}\binom{m}{l} \times \binom{n}{j}\binom{m}{k}$ matrix N_1, with rows indexed by $\binom{[n]}{j} \times \binom{[m]}{l}$, columns indexed by $\binom{[n]}{j} \times \binom{[m]}{k}$ and with entry in row (A, B), column (C, D) equal to 1 if $A = C$ and $B \supseteq D$ and equal to 0 otherwise.

Consider the matrices N and N_1N_2, whose rows and columns are indexed by the same sets. Let $(A, B) \in \binom{[n]}{j} \times \binom{[m]}{l}$, $C \in \binom{[n]}{i}$, $D \in \binom{[m]}{k}$. Then

$$\text{entry in row}(A, B), \quad \text{column } C \text{ of } N_1N_2 = \begin{cases} \binom{l}{k} & \text{if } A \supseteq C, \\ 0 & \text{if } A \not\supseteq C, \end{cases}$$

$$\text{entry in row}(A, B), \quad \text{column } D \text{ of } N_1N_2 = \begin{cases} 1 & \text{if } B \supseteq D, \\ 0 & \text{if } B \not\supseteq D. \end{cases}$$

It follows that every column of N_1N_2 is a nonzero multiple of the corresponding column of N. Thus N and N_1N_2 have the same ranks. We shall now compute the rank of N_1N_2.

Note that N_1 is the Kronecker product $I_n(j, j) \otimes I_m(l, k)$. Now, the rank of a Kronecker product is the product of the ranks and thus, by Theorem 4.2, $\operatorname{rank}(N_1) = \binom{n}{j} \times \binom{m}{k}$, i.e., N_1 has full column rank. We will now show that $\operatorname{rank}(N_2) = \binom{n}{i} + \binom{m}{k} - 1$. This will prove the theorem.

Put $t = \binom{m}{k}$. Let $X_1, X_2, \ldots, X_t$ be a list of the elements of $\binom{[m]}{k}$. List the columns of N_2 in the following order: first the elements of $\binom{[n]}{i}$ (in any order) followed by the elements $\binom{[m]}{k}$ in the order $X_1, X_2, \ldots, X_t$. List the elements of

$\binom{[n]}{j} \times \binom{[m]}{k}$ in the following order: first the elements of $\binom{[n]}{j} \times \{X_1\}$, followed by the elements of $\binom{[n]}{j} \times \{X_2\}$, and so on. With this listing N_2 has the block form shown below.

$$
\begin{array}{cc}
 & \begin{matrix} \binom{[n]}{i} & \quad X_1 & X_2 & \qquad\qquad & X_t \end{matrix} \\
\begin{matrix} \binom{[n]}{j} \times \{X_1\} \\[2em] \binom{[n]}{j} \times \{X_2\} \\[2em] \vdots \\[2em] \binom{[n]}{j} \times \{X_t\} \end{matrix} &
\begin{bmatrix}
\boxed{I_n(j,i)} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\boxed{I_n(j,i)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
 & \vdots & \vdots & & \vdots \\
\boxed{I_n(j,i)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}
\end{bmatrix}
\end{array}.
$$

By Theorem 4.2, it now follows that the first $\binom{n}{i}$ columns together with the last $t-1$ columns of N_2 are linearly independent. Thus $\operatorname{rank}(N_2) \geqslant \binom{n}{i} + \binom{m}{k} - 1$. Since every row of N_2 is orthogonal to the vector $(-1, -1, \dots, -1, \binom{j}{i}, \dots, \binom{j}{i})$ (-1 is repeated $\binom{n}{i}$ times and $\binom{j}{i}$ is repeated $\binom{m}{k}$ times), we have $\operatorname{rank}(N_2) = \binom{n}{i} + \binom{m}{k} - 1$. $\square$

For the purposes of Section 3 we only need the following special case of the result above.

Theorem 4.5. *Let S, T be finite, disjoint, nonempty sets and let p, q be positive integers. Let M be the matrix with columns indexed by $S \cup T$, rows indexed by $\binom{S}{q} \times \binom{T}{p}$, and with entry in row (A, B), column x equal to 1 if $x \in A \cup B$ and equal to 0 otherwise. Then the rank of M is $|S| + |T| - 1$ if and only if one of the following conditions holds*:

(i) $p = q = 1,\ |S| = |T| = 1.$
(ii) $p = 1,\ |T| = 1,\ |S| = q + 1.$
(iii) $q = 1,\ |S| = 1,\ |T| = p + 1.$
(iv) $|T| \geqslant p + 1,\ |S| \geqslant q + 1.$

Proof. (*Only if*) Let $s = |S|$ and $t = |T|$. Then M has $\binom{s}{q}\binom{t}{p}$ rows and $s + t$ columns. If $s < q$ or $t < p$, then M has no rows and the rank of M cannot be $s + t - 1 \geqslant 1$. So $s \geqslant q,\ t \geqslant p$. Assume that condition (iv) does not hold. Then there are the following three possibilities:

(a) $s = q, t = p$: Since $\binom{s}{q}\binom{t}{p} = 1$, we have $\mathrm{rank}(M) = 1$ and thus $s = q = t = p = 1$.

(b) $s = q + 1,\ t = p$: Since $\binom{s}{q}\binom{t}{p} = q + 1$ and M has $p + q + 1$ columns, we have $\mathrm{rank}(M) = s + t - 1 = p + q$ only if $t = p = 1$.

(c) $s = q,\ t = p + 1$: Just as in case (b), we have $s = q = 1$.

(*If*) Follows from Theorem 4.4. □

5. Facets of $D_4(3)$, $D_5(3)$, and $D_6(3)$

In this section we give the complete list of facets of $D_4(3)$, $D_5(3)$, and $D_6(3)$, generated using the package PORTA [7]. Using Theorem 3.8 we can give a proof, independent of PORTA, that these are all the facets of $D_4(3)$, $D_5(3)$, and $D_6(3)$. As the proof is somewhat long and tedious, we omit it.

Facets of $D_4(3)$:

1. $2\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 0,\ S, T \subseteq [4],\ S \cap T = \emptyset,\ |S| = 1,\ |T| = 3.$
2. $\sum_{i\in S} x_i - 2\sum_{i\in T} x_i \leqslant 1,\ S, T \subseteq [4],\ S \cap T = \emptyset,\ |S| = 3,\ |T| = 1.$

Facets of $D_5(3)$:

1. $x_i \geqslant 0,\ i = 1, 2, 3, 4, 5.$
2. $x_i \leqslant 6,\ i = 1, 2, 3, 4, 5.$
3. $\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 3,\ S, T \subseteq [5],\ S \cap T = \emptyset, |S| = 1,\ |T| = 1.$
4. $\sum_{i\in S} x_i - 2\sum_{i\in T} x_i \leqslant 3,\ S, T \subseteq [5],\ S \cap T = \emptyset,\ |S| = 3,\ |T| = 2.$
5. $2\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 9,\ S, T \subseteq [5],\ S \cap T = \emptyset,\ |S| = 2,\ |T| = 3.$
6. $2\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 0,\ S, T \subseteq [5],\ S \cap T = \emptyset,\ |S| = 1,\ |T| = 4.$
7. $\sum_{i\in S} x_i - 2\sum_{i\in T} x_i \leqslant 12,\ S, T \subseteq [5],\ S \cap T = \emptyset,\ |S| = 4,\ |T| = 1.$

Facets of $D_6(3)$:

1. $x_i \geqslant 0,\ i = 1, 2, 3, 4, 5, 6.$
2. $x_i \leqslant 10,\ i = 1, 2, 3, 4, 5, 6.$
3. $\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 6,\ S, T \subseteq [6],\ S \cap T = \emptyset,\ |S| = 1,\ |T| = 1.$
4. $\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 13,\ S, T \subseteq [6],\ S \cap T = \emptyset,\ |S| = 2,\ |T| = 1.$
5. $\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 3,\ S, T \subseteq [6],\ S \cap T = \emptyset,\ |S| = 1,\ |T| = 2.$
6. $\sum_{i\in S} x_i - 2\sum_{i\in T} x_i \leqslant 30,\ S, T \subseteq [6],\ S \cap T = \emptyset,\ |S| = 5,\ |T| = 1.$

7. $2\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 0,\ S, T \subseteq [6],\ S\cap T = \emptyset,\ |S| = 1,\ |T| = 5.$
8. $\sum_{i\in S} x_i - 2\sum_{i\in T} x_i \leqslant 12,\ S, T \subseteq [6],\ S\cap T = \emptyset,\ |S| = 4,\ |T| = 2.$
9. $\sum_{i\in S_1} x_i + 4\sum_{i\in S_2} x_i - 2\sum_{i\in T} x_i \leqslant 39,\ S_1, S_2, T \subseteq [6],$
   S_1, S_2, T disjoint, $|S_1| = 3,\ |S_2| = 1,\ |T| = 2.$
10. $2\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 12,\ S, T \subseteq [6],\ S\cap T = \emptyset, |S| = 2, |T| = 4.$
11. $2\sum_{i\in S} x_i - \sum_{i\in T_1} x_i - 4\sum_{i\in T_2} x_i \leqslant 9,\ S, T_1, T_2 \subseteq [6],$
    S, T_1, T_2 disjoint, $|S| = 2,\ |T_1| = 3,\ |T_2| = 1.$
12. $2\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant 33,\ S, T \subseteq [6],\ S\cap T = \emptyset,\ |S| = 3,\ |T| = 3.$
13. $\sum_{i\in S} x_i - 2\sum_{i\in T} x_i \leqslant 3,\ S, T \subseteq [6],\ S\cap T = \emptyset,\ |S| = 3,\ |T| = 3.$
14. $2\sum_{i\in S_1} x_i + 5\sum_{i\in S_2} x_i - \sum_{i\in T_1} x_i - 4\sum_{i\in T_2} x_i \leqslant 48, \quad S_1, S_2, T_1T_2 \subseteq [6],$
    S_1, S_2, T_1, T_2 disjoint, $|S_1| = 2,\ |S_2| = 1,\ |T_1| = 2,\ |T_2| = 1.$
15. $\sum_{i\in S_1} x_i + 4\sum_{i\in S_2} x_i - 2\sum_{i\in T_1} x_i - 5\sum_{i\in T_2} x_i \leqslant 18, \quad S_1, S_2, T_1, T_2 \subseteq [6],$
    S_1, S_2, T_1, T_2 disjoint, $|S_1| = 2,\ |S_2| = 1,\ |T_1| = 2,\ |T_2| = 1.$

Finally, we would like to state the following problems: for fixed r, find combinatorial polynomial time algorithms to test whether a nonnegative integral vector is an r-threshold sequence and to test whether a nonnegative rational vector belongs to $D_n(r)$. Since, for fixed r, we can optimize linear functions over $D_n(r)$ in polynomial time, it follows that, using the ellipsoid method (see [12,14]), we can solve these two problems in polynomial time. A combinatorial solution would significantly enhance our understanding of hypergraph degree sequences.

Acknowledgements

We would like to thank R.B. Bapat, Prof. N.R. Achuthan, and the referee for several useful suggestions on the write-up. We are most grateful to Prof. U.N. Peled for his support and encouragement.

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